CHƯƠNG 1: CÔNG THỨC LƯỢNG GIÁC

I. Định nghĩa

Trên mặt phẳng Oxy cho đường tròn lượng giác tâm O bán kính R=1 và điểm M trên đường tròn lượng giác mà sđ $\overset{\frown}{AM} = \beta$ với $0 \le \beta \le 2\pi$

Đặt $\alpha = \beta + k2\pi, k \in Z$

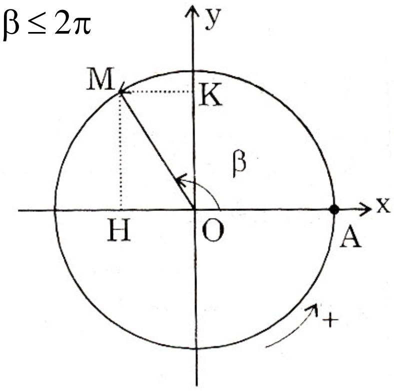
Ta định nghĩa:

$\sin\alpha = \overline{OK}$

$\cos\alpha = \overline{OH}$

$tg\alpha = \dfrac{\sin\alpha}{\cos\alpha}$ với $\cos\alpha \ne 0$

$\cot g\alpha = \dfrac{\cos\alpha}{\sin\alpha}$ với $\sin\alpha \ne 0$



II. Bảng giá trị lượng giác của một số cung (hay góc) đặc biệt

Góc α / Giá trị	$0(0^o)$	$\frac{\pi}{6}(30^o)$	$\frac{\pi}{4}(45^o)$	$\frac{\pi}{3}(60^o)$	$\frac{\pi}{2}(90^o)$
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$tg\alpha$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\parallel$
$\cot g\alpha$	$\parallel$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

III. Hệ thức cơ bản

$\sin^2\alpha + \cos^2\alpha = 1$

$1 + tg^2\alpha = \dfrac{1}{\cos^2\alpha}$ với $\alpha \ne \dfrac{\pi}{2} + k\pi\,(k \in Z)$

$t + \cot g^2 = \dfrac{1}{\sin^2\alpha}$ với $\alpha \ne k\pi\,(k \in Z)$

IV. Cung liên kết (Cách nhớ: cos đối, sin bù, tang sai π; phụ chéo)

a. Đối nhau: α và $-\alpha$

$\sin(-\alpha) = -\sin\alpha$

$\cos(-\alpha) = \cos\alpha$

$tg(-\alpha) = -tg(\alpha)$

$\cot g(-\alpha) = -\cot g(\alpha)$

b. Bù nhau: α và $\pi - \alpha$

$$\sin(\pi-\alpha)=\sin\alpha$$
$$\cos(\pi-\alpha)=-\cos\alpha$$
$$tg(\pi-\alpha)=-tg\alpha$$
$$cotg(\pi-\alpha)=-cotg\alpha$$

c. Sai nhau π: α và $\pi+\alpha$

$$\sin(\pi+\alpha)=-\sin\alpha$$
$$\cos(\pi+\alpha)=-\cos\alpha$$
$$tg(\pi+\alpha)=tg\alpha$$
$$cotg(\pi+\alpha)=cotg\alpha$$

d. Phụ nhau: α và $\frac{\pi}{2}-\alpha$

$$\sin\left(\frac{\pi}{2}-\alpha\right)=\cos\alpha$$
$$\cos\left(\frac{\pi}{2}-\alpha\right)=\sin\alpha$$
$$tg\left(\frac{\pi}{2}-\alpha\right)=cotg\alpha$$
$$cotg\left(\frac{\pi}{2}-\alpha\right)=tg\alpha$$

e.Sai nhau $\frac{\pi}{2}$: α và $\frac{\pi}{2}+\alpha$

$$\sin\left(\frac{\pi}{2}+\alpha\right)=\cos\alpha$$
$$\cos\left(\frac{\pi}{2}+\alpha\right)=-\sin\alpha$$
$$tg\left(\frac{\pi}{2}+\alpha\right)=-cotg\alpha$$
$$cotg\left(\frac{\pi}{2}+\alpha\right)=-tg\alpha$$

f.

$$\sin(x+k\pi)=(-1)^k\sin x, k\in Z$$
$$\cos(x+k\pi)=(-1)^k\cos x, k\in Z$$
$$tg(x+k\pi)=tgx, k\in Z$$
$$\cot g(x+k\pi)=\cot gx$$

V. Công thức cộng

$$\sin(a\pm b)=\sin a\cos b\pm\sin b\cos a$$
$$\cos(a\pm b)=\cos a\cos b\mp\sin a\sin b$$
$$tg(a\pm b)=\frac{tga\pm tgb}{1\mp tgatgb}$$

VI. Công thức nhân đôi

$$\sin 2a=2\sin a\cos a$$
$$\cos 2a=\cos^2 a-\sin^2 a=1-2\sin^2 a=2\cos^2 a-1$$
$$tg2a=\frac{2tga}{1-tg^2a}$$
$$\cot g2a=\frac{\cot g^2a-1}{2\cot ga}$$

VII. Công thức nhân ba:

$$\sin 3a=3\sin a-4\sin^3 a$$
$$\cos 3a=4\cos^3 a-3\cos a$$

VIII. Công thức hạ bậc:

$$\sin^2 a=\frac{1}{2}(1-\cos 2a)$$
$$\cos^2 a=\frac{1}{2}(1+\cos 2a)$$
$$tg^2a=\frac{1-\cos 2a}{1+\cos 2a}$$

IX. Công thức chia đôi

Đặt $t=tg\frac{a}{2}$ (với $a\neq\pi+k2\pi$)

$$\sin a = \frac{2t}{1+t^2}$$
$$\cos a = \frac{1-t^2}{1+t^2}$$
$$tga = \frac{2t}{1-t^2}$$

X. Công thức biến đổi tổng thành tích

$$\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{a-b}{2}$$
$$\cos a - \cos b = -2\sin\frac{a+b}{2}\sin\frac{a-b}{2}$$
$$\sin a + \sin b = 2\cos\frac{a+b}{2}\sin\frac{a-b}{2}$$
$$\sin a - \sin b = 2\cos\frac{a+b}{2}\sin\frac{a-b}{2}$$
$$tga \pm tgb = \frac{\sin(a \pm b)}{\cos a \cos b}$$
$$\cot ga \pm \cot gb = \frac{\sin(b \pm a)}{\sin a.\sin b}$$

XI. Công thức biển đổi tích thành tổng

$$\cos a.\cos b = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right]$$
$$\sin a.\sin b = \frac{-1}{2}\left[\cos(a+b) - \cos(a-b)\right]$$
$$\sin a.\cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$$

Bài 1: Chứng minh $\frac{\sin^4 a + \cos^4 a - 1}{\sin^6 a + \cos^6 a - 1} = \frac{2}{3}$

Ta có:

$$\sin^4 a + \cos^4 a - 1 = \left(\sin^2 a + \cos^2 a\right)^2 - 2\sin^2 a\cos^2 a - 1 = -2\sin^2 a\cos^2 a$$

Và:

$$\sin^6 a + \cos^6 a - 1 = \left(\sin^2 a + \cos^2 a\right)\left(\sin^4 a - \sin^2 a\cos^2 a + \cos^4 a\right) - 1$$
$$= \sin^4 a + \cos^4 a - \sin^2 a\cos^2 a - 1$$
$$= \left(1 - 2\sin^2 a\cos^2 a\right) - \sin^2 a\cos^2 a - 1$$
$$= -3\sin^2 a\cos^2 a$$

Do đó: $\dfrac{\sin^4 a+\cos^4 a-1}{\sin^6 a+\cos^6 a-1}=\dfrac{-2\sin^2 a\cos^2 a}{-3\sin^2 a\cos^2 a}=\dfrac{2}{3}$

Bài 2: Rút gọn biểu thức $A=\dfrac{1+\cos x}{\sin x}=\left[1+\dfrac{(1-\cos x)^2}{\sin^2 x}\right]$

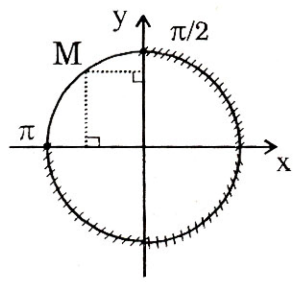
Tính giá trị A nếu $\cos x=-\dfrac{1}{2}$ và $\dfrac{\pi}{2}<x<\pi$

Ta có: $A=\dfrac{1+\cos x}{\sin x}\left(\dfrac{\sin^2 x+1-2\cos x+\cos^2 x}{\sin^2 x}\right)$

$\Leftrightarrow A=\dfrac{1+\cos x}{\sin x}.\dfrac{2(1-\cos x)}{\sin^2 x}$

$\Leftrightarrow A=\dfrac{2(1-\cos^2 x)}{\sin^3 x}=\dfrac{2\sin^2 x}{\sin^3 x}=\dfrac{2}{\sin x}$ (với $\sin x\neq 0$)

Ta có: $\sin^2 x=1-\cos^2 x=1-\dfrac{1}{4}=\dfrac{3}{4}$



Do: $\dfrac{\pi}{2}<x<\pi$ nên $\sin x>0$

Vậy $\sin x=\dfrac{\sqrt{3}}{2}$

Do đó $A=\dfrac{2}{\sin x}=\dfrac{4}{\sqrt{3}}=\dfrac{4\sqrt{3}}{3}$

Bài 3: Chứng minh các biểu thức sau đây không phụ thuộc x:

a. $A=2\cos^4 x-\sin^4 x+\sin^2 x\cos^2 x+3\sin^2 x$

b. $B=\dfrac{2}{\text{tg}x-1}+\dfrac{\cot gx+1}{\cot gx-1}$

a. Ta có:

$A=2\cos^4 x-\sin^4 x+\sin^2 x\cos^2 x+3\sin^2 x$

$\Leftrightarrow A=2\cos^4 x-(1-\cos^2 x)^2+(1-\cos^2 x)\cos^2 x+3(1-\cos^2 x)$

$\Leftrightarrow A=2\cos^4 x-(1-2\cos^2 x+\cos^4 x)+\cos^2 x-\cos^4 x+3-3\cos^2 x$

$\Leftrightarrow A=2$ (không phụ thuộc x)

b. Với điều kiện $\sin x.\cos x\neq 0, \text{tg}x\neq 1$

Ta có: $B=\dfrac{2}{\text{tg}x-1}+\dfrac{\cot gx+1}{\cot gx-1}$

$$\Leftrightarrow B=\frac{2}{tgx-1}+\frac{\frac{1}{tgx}+1}{\frac{1}{tgx}-1}=\frac{2}{tgx-1}+\frac{1+tgx}{1-tgx}$$

$$\Leftrightarrow B=\frac{2-(1-tgx)}{tgx-1}=\frac{1-tgx}{tgx-1}=-1 \text{ (không phụ thuộc vào x)}$$

Bài 4: Chứng minh

$$\frac{1+\cos a}{2\sin a}\left[1-\frac{(1-\cos a)^2}{\sin^2 a}\right]+\frac{\cos^2 b-\sin^2 c}{\sin^2 b\sin^2 c}-\cot g^2 b\cot g^2 c=\cot ga-1$$

Ta có:

$$* \ \frac{\cos^2 b-\sin^2 c}{\sin^2 b.\sin^2 c}-\cot g^2 b.\cot g^2 c$$

$$=\frac{\cot g^2 b}{\sin^2 c}-\frac{1}{\sin^2 b}-\cot g^2 b\cot g^2 c$$

$$=\cot g^2 b\left(1+\cot g^2 c\right)-\left(1+\cot g^2 b\right)-\cot g^2 b\cot g^2 c=-1 \quad (1)$$

$$* \ \frac{1+\cos a}{2\sin a}\left[1-\frac{(1-\cos a)^2}{\sin^2 a}\right]$$

$$=\frac{1+\cos a}{2\sin a}\left[1-\frac{(1-\cos a)^2}{1-\cos^2 a}\right]$$

$$=\frac{1+\cos a}{2\sin a}\left[1-\frac{1-\cos a}{1+\cos a}\right]$$

$$=\frac{1+\cos a}{2\sin a}.\frac{2\cos a}{1+\cos a}=\cot ga \quad (2)$$

Lấy (1) + (2) ta được điều phải chứng minh xong.

Bài 5: Cho ΔABC tùy ý với ba góc đều là nhọn.
Tìm giá trị nhỏ nhất của $P=tgA.tgB.tgC$

Ta có: $A+B=\pi-C$

Nên: $tg(A+B)=-tgC$

$$\Leftrightarrow \frac{tgA+tgB}{1-tgA.tgB}=-tgC$$

$\Leftrightarrow tgA+tgB=-tgC+tgA.tgB.tgC$

Vậy: $P=tgA.tgB.tgC=tgA+tgB+tgC$

Áp dụng bất đẳng thức Cauchy cho ba số dương tgA, tgB, tgC ta được $tgA + tgB + tgC \geq 3\sqrt[3]{tgA.tgB.tgC}$

$\Leftrightarrow P \geq 3\sqrt[3]{P}$

$\Leftrightarrow \sqrt[3]{P^2} \geq 3$

$\Leftrightarrow P \geq 3\sqrt{3}$

Dấu "=" xảy ra $\Leftrightarrow \begin{cases} tgA = tgB = tgC \\ 0 < A,B,C < \dfrac{\pi}{2} \end{cases} \Leftrightarrow A = B = C = \dfrac{\pi}{3}$

Do đó: $MinP = 3\sqrt{3} \Leftrightarrow A = B = C = \dfrac{\pi}{3}$

Bài 6 : Tìm giá trị lớn nhất và nhỏ nhất của

a/ $y = 2\sin^8 x + \cos^4 2x$

b/ $y = \sqrt[4]{\sin x} - \sqrt{\cos x}$

a/ Ta có : $y = 2\left(\dfrac{1-\cos 2x}{2}\right)^4 + \cos^4 2x$

Đặt $t = \cos 2x$ với $-1 \leq t \leq 1$ thì

$y = \dfrac{1}{8}(1-t)^4 + t^4$

$\Rightarrow y' = -\dfrac{1}{2}(1-t)^3 + 4t^3$

Ta có : $y' = 0 \Leftrightarrow (1-t)^3 = 8t^3$

$\Leftrightarrow 1 - t = 2t$

$\Leftrightarrow t = \dfrac{1}{3}$

Ta có y(1) = 1; y(-1) = 3; $y\left(\dfrac{1}{3}\right) = \dfrac{1}{27}$

Do đó : $\underset{x\in\mathbb{R}}{\text{Max}}\, y = 3$ và $\underset{x\in\mathbb{R}}{\text{Min}}\, y = \dfrac{1}{27}$

b/ Do điều kiện : $\sin x \geq 0$ và $\cos x \geq 0$ nên miền xác định

$D = \left[k2\pi, \dfrac{\pi}{2} + k2\pi\right]$ với $k \in \mathbb{Z}$

Đặt $t = \sqrt{\cos x}$ với $0 \leq t \leq 1$ thì $t^4 = \cos^2 x = 1 - \sin^2 x$

Nên $\sin x = \sqrt{1 - t^4}$

Vậy $y = \sqrt[8]{1 - t^4} - t$ trên $D' = [0,1]$

Thì $y' = \dfrac{-t^3}{2.\sqrt[8]{(1-t^4)^7}} - 1 < 0 \quad \forall t \in [0;1)$

Nên y giảm trên [0, 1]. Vậy : $\max\limits_{x\in D} y = y(0) = 1, \ \min\limits_{x\in D} y = y(1) = -1$

Bài 7: Cho hàm số $y = \sqrt{\sin^4 x + \cos^4 x - 2m \sin x \cos x}$
Tìm giá trị m để y xác định với mọi x

Xét $f(x) = \sin^4 x + \cos^4 x - 2m \sin x \cos x$

$f(x) = (\sin^2 x + \cos^2 x)^2 - m\sin 2x - 2\sin^2 x \cos^2 x$

$f(x) = 1 - \frac{1}{2}\sin^2 2x - m\sin 2x$

Đặt : $t = \sin 2x$ với $t \in [-1,1]$

y xác định $\forall x \Leftrightarrow f(x) \geq 0 \forall x \in R$

$\Leftrightarrow 1 - \frac{1}{2}t^2 - mt \geq 0 \quad \forall t \in [-1,1]$

$\Leftrightarrow g(t) = t^2 + 2mt - 2 \leq 0 \quad \forall t \in [-1,1]$

Do $\Delta' = m^2 + 2 > 0 \quad \forall m$ nên g(t) có 2 nghiệm phân biệt t_1, t_2

Lúc đó

t		t_1		t_2
g(t)	+	0	-	0

Do đó : yêu cầu bài toán $\Leftrightarrow t_1 \leq -1 < 1 \leq t_2$

$\Leftrightarrow \begin{cases} 1g(-1) \leq 0 \\ 1g(1) \leq 0 \end{cases} \Leftrightarrow \begin{cases} -2m - 1 \leq 0 \\ 2m - 1 \leq 0 \end{cases}$

$\Leftrightarrow \begin{cases} m \geq \frac{-1}{2} \\ m \leq \frac{1}{2} \end{cases} \Leftrightarrow -\frac{1}{2} \leq m \leq \frac{1}{2}$

Cách khác :

$g(t) = t^2 + 2mt - 2 \leq 0 \quad \forall t \in [-1,1]$

$\Leftrightarrow \max_{t\in[-1,1]} g(t) \leq 0 \Leftrightarrow \max\{g(-1), g(1)\} \leq 0$

$\Leftrightarrow \max\{-2m-1), -2m+1)\} \leq 0 \Leftrightarrow \begin{cases} m \geq \frac{-1}{2} \\ m \leq \frac{1}{2} \end{cases}$

$\Leftrightarrow -\frac{1}{2} \leq m \leq \frac{1}{2}$

Bài 8 : Chứng minh $A = \sin^4 \frac{\pi}{16} + \sin^4 \frac{3\pi}{16} + \sin^4 \frac{5\pi}{16} + \sin^4 \frac{7\pi}{16} = \frac{3}{2}$

Ta có : $\sin\frac{7\pi}{16} = \sin\left(\frac{\pi}{2} - \frac{\pi}{16}\right) = \cos\frac{\pi}{16}$

$\sin\frac{5\pi}{16} = \cos\left(\frac{\pi}{2} - \frac{5\pi}{16}\right) = \cos\frac{3\pi}{16}$

Mặt khác : $\sin^4\alpha + \cos^4\alpha = \left(\sin^2\alpha + \cos^2\alpha\right)^2 - 2\sin^2\alpha\cos^2\alpha$

$$= 1 - 2\sin^2\alpha\cos^2\alpha$$

$$= 1 - \frac{1}{2}\sin^2 2\alpha$$

Do đó : $A = \sin^4\frac{\pi}{16} + \sin^4\frac{7\pi}{16} + \sin^4\frac{3\pi}{16} + \sin^4\frac{5\pi}{16}$

$$= \left(\sin^4\frac{\pi}{16} + \cos^4\frac{\pi}{16}\right) + \left(\sin^4\frac{3\pi}{16} + \cos^4\frac{3\pi}{16}\right)$$

$$= \left(1 - \frac{1}{2}\sin^2\frac{\pi}{8}\right) + \left(1 - \frac{1}{2}\sin^2\frac{3\pi}{8}\right)$$

$$= 2 - \frac{1}{2}\left(\sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8}\right)$$

$$= 2 - \frac{1}{2}\left(\sin^2\frac{\pi}{8} + \cos^2\frac{\pi}{8}\right) \quad \left(\text{do } \sin\frac{3\pi}{8} = \cos\frac{\pi}{8}\right)$$

$$= 2 - \frac{1}{2} = \frac{3}{2}$$

Bài 9 : Chứng minh : $16\sin 10^o.\sin 30^o.\sin 50^o.\sin 70^o = 1$

Ta có : $A = \dfrac{A\cos 10^o}{\cos 10^o} = \dfrac{1}{\cos 10^o}(16\sin 10^o\cos 10^o)\sin 30^o.\sin 50^o.\sin 70^o$

$$\Leftrightarrow A = \frac{1}{\cos 10^o}\left(8\sin 20^o\right)\left(\frac{1}{2}\right)\cos 40^o.\cos 20^o$$

$$\Leftrightarrow A = \frac{1}{\cos 10^o}\left(4\sin 20^0\cos 20^o\right).\cos 40^o$$

$$\Leftrightarrow A = \frac{1}{\cos 10^o}\left(2\sin 40^o\right)\cos 40^o$$

$$\Leftrightarrow A = \frac{1}{\cos 10^o}\sin 80^o = \frac{\cos 10^o}{\cos 10^o} = 1$$

Bài 10 : Cho ΔABC. Chứng minh : $\operatorname{tg}\frac{A}{2}\operatorname{tg}\frac{B}{2} + \operatorname{tg}\frac{B}{2}\operatorname{tg}\frac{C}{2} + \operatorname{tg}\frac{C}{2}\operatorname{tg}\frac{A}{2} = 1$

Ta có : $\dfrac{A+B}{2} = \dfrac{\pi}{2} - \dfrac{C}{2}$

Vậy : $\operatorname{tg}\dfrac{A+B}{2} = \operatorname{cotg}\dfrac{C}{2}$

$$\Leftrightarrow \frac{\operatorname{tg}\frac{A}{2} + \operatorname{tg}\frac{B}{2}}{1 - \operatorname{tg}\frac{A}{2}.\operatorname{tg}\frac{B}{2}} = \frac{1}{\operatorname{tg}\frac{C}{2}}$$

$$\Leftrightarrow \left[\operatorname{tg}\frac{A}{2} + \operatorname{tg}\frac{B}{2}\right]\operatorname{tg}\frac{C}{2} = 1 - \operatorname{tg}\frac{A}{2}\operatorname{tg}\frac{B}{2}$$

$$\Leftrightarrow \ tg\frac{A}{2}tg\frac{C}{2} + tg\frac{B}{2}tg\frac{C}{2} + tg\frac{A}{2}tg\frac{B}{2} = 1$$

Bài 11 : Chứng minh : $8 + 4tg\frac{\pi}{8} + 2tg\frac{\pi}{16} + tg\frac{\pi}{32} = \cot g\frac{\pi}{32}(*)$

Ta có : $(*) \Leftrightarrow 8 = \cot g\frac{\pi}{32} - tg\frac{\pi}{32} - 2tg\frac{\pi}{16} - 4tg\frac{\pi}{8}$

Mà : $\cot ga - tga = \frac{\cos a}{\sin a} - \frac{\sin a}{\cos a} = \frac{\cos^2 a - \sin^2 a}{\sin a\cos a}$

$$= \frac{\cos 2a}{\frac{1}{2}\sin 2a} = 2\cot g2a$$

Do đó :

$$(*) \Leftrightarrow \left[\cot g\frac{\pi}{32} - tg\frac{\pi}{32}\right] - 2tg\frac{\pi}{16} - 4tg\frac{\pi}{8} = 8$$

$$\Leftrightarrow \left[2\cot g\frac{\pi}{16} - 2tg\frac{\pi}{16}\right] - 4tg\frac{\pi}{8} = 8$$

$$\Leftrightarrow 4\cot g\frac{\pi}{8} - 4tg\frac{\pi}{8} = 8$$

$$\Leftrightarrow 8\cot g\frac{\pi}{4} = 8 \ \text{(hiển nhiên đúng)}$$

Bài :12 : Chứng minh :

a/ $\cos^2 x + \cos^2\left(\frac{2\pi}{3} + x\right) + \cos^2\left(\frac{2\pi}{3} - x\right) = \frac{3}{2}$

b/ $\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \frac{1}{\sin 8x} + \frac{1}{\sin 16x} = \cot gx - \cot g16x$

a/ Ta có : $\cos^2 x + \cos^2\left(\frac{2\pi}{3} + x\right) + \cos^2\left(\frac{2\pi}{3} - x\right)$

$$= \frac{1}{2}(1 + \cos 2x) + \frac{1}{2}\left[1 + \cos\left(2x + \frac{4\pi}{3}\right)\right] + \frac{1}{2}\left[1 + \cos\left(\frac{4\pi}{3} - 2x\right)\right]$$

$$= \frac{3}{2} + \frac{1}{2}\left[\cos 2x + \cos\left(2x + \frac{4\pi}{3}\right) + \cos\left(\frac{4\pi}{3} - 2x\right)\right]$$

$$= \frac{3}{2} + \frac{1}{2}\left[\cos 2x + 2\cos 2x\cos\frac{4\pi}{3}\right]$$

$$= \frac{3}{2} + \frac{1}{2}\left[\cos 2x + 2\cos 2x\left(-\frac{1}{2}\right)\right]$$

$$= \frac{3}{2}$$

b/ Ta có : $\cot ga - \cot gb = \frac{\cos a}{\sin a} - \frac{\cos b}{\sin b} = \frac{\sin b\cos a - \sin a\cos b}{\sin a\sin b}$

$$= \frac{\sin(b-a)}{\sin a \sin b}$$

Do đó : $\cot gx - \cot g2x = \dfrac{\sin(2x-x)}{\sin x \sin 2x} = \dfrac{1}{\sin 2x}(1)$

$$\cot g2x - \cot g4x = \frac{\sin(4x-2x)}{\sin 2x \sin 4x} = \frac{1}{\sin 4x}(2)$$

$$\cot g4x - \cot g8x = \frac{\sin(8x-4x)}{\sin 4x \sin 8x} = \frac{1}{\sin 8x}(3)$$

$$\cot g8x - \cot g16x = \frac{\sin(16x-8x)}{\sin 16x \sin 8x} = \frac{1}{\sin 16x}(4)$$

Lấy (1) + (2) + (3) + (4) ta được

$$\cot gx - \cot g16x = \frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \frac{1}{\sin 8x} + \frac{1}{\sin 16x}$$

Bài 13 : Chứng minh : $8\sin^3 18^0 + 8\sin^2 18^0 = 1$

Ta có: $\sin 18^0 = \cos 72^0$

$\Leftrightarrow \sin 18^0 = 2\cos^2 36^0 - 1$

$\Leftrightarrow \sin 18^0 = 2(1 - 2\sin^2 18^0)^2 - 1$

$\Leftrightarrow \sin 18^0 = 2(1 - 4\sin^2 18^0 + 4\sin^4 18^0) - 1$

$\Leftrightarrow 8\sin^4 18^0 - 8\sin^2 18^0 - \sin 18^0 + 1 = 0$ (1)

$\Leftrightarrow (\sin 18^0 - 1)(8\sin^3 18^0 + 8\sin^2 18^0 - 1) = 0$

$\Leftrightarrow 8\sin^3 18^0 + 8\sin^2 18^0 - 1 = 0$ (do $0 < \sin 18^0 < 1$)

Cách khác :

Chia 2 vế của (1) cho ($\sin 18^0 - 1$) ta có

(1) $\Leftrightarrow 8\sin^2 18^0 (\sin 18^0 + 1) - 1 = 0$

Bài 14 : Chứng minh :

a/ $\sin^4 x + \cos^4 x = \dfrac{1}{4}(3 + \cos 4x)$

b/ $\sin 6x + \cos 6x = \dfrac{1}{8}(5 + 3\cos 4x)$

c/ $\sin^8 x + \cos^8 x = \dfrac{1}{64}(35 + 28\cos 4x + \cos 8x)$

a/ Ta có: $\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x$

$$= 1 - \frac{2}{4}\sin^2 2x$$

$$= 1 - \frac{1}{4}(1 - \cos 4x)$$

$$= \frac{3}{4} + \frac{1}{4}\cos 4x$$

b/ Ta có : sin6x + cos6x

$$= (\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x)$$

$$= \left(\sin^4 x + \cos^4 x\right) - \frac{1}{4}\sin^2 2x$$

$$= \left(\frac{3}{4} + \frac{1}{4}\cos 4x\right) - \frac{1}{8}\left(1 - \cos 4x\right) \quad \text{(do kết quả câu a)}$$

$$= \frac{3}{8}\cos 4x + \frac{5}{8}$$

c/ Ta có : $\sin^8 x + \cos^8 x = \left(\sin^4 x + \cos^4 x\right)^2 - 2\sin^4 x\cos^4 x$

$$= \frac{1}{16}\left(3 + \cos 4x\right)^2 - \frac{2}{16}\sin^4 2x$$

$$= \frac{1}{16}\left(9 + 6\cos 4x + \cos^2 4x\right) - \frac{1}{8}\left[\frac{1}{2}\left(1 - \cos 4x\right)\right]^2$$

$$= \frac{9}{16} + \frac{3}{8}\cos 4x + \frac{1}{32}\left(1 + \cos 8x\right) - \frac{1}{32}\left(1 - 2\cos 4x + \cos^2 4x\right)$$

$$= \frac{9}{16} + \frac{3}{8}\cos 4x + \frac{1}{32}\cos 8x + \frac{1}{16}\cos 4x - \frac{1}{64}\left(1 + \cos 8x\right)$$

$$= \frac{35}{64} + \frac{7}{16}\cos 4x + \frac{1}{64}\cos 8x$$

Bài 15 : Chứng minh : $\sin 3x.\sin^3 x + \cos 3x.\cos^3 x = \cos^3 2x$

Cách 1:

Ta có : $\sin 3x.\sin^3 x + \cos 3x.\cos^3 x = \cos^3 2x$

$$= \left(3\sin x - 4\sin^3 x\right)\sin^3 x + \left(4\cos^3 x - 3\cos x\right)\cos^3 x$$

$$= 3\sin^4 x - 4\sin^6 x + 4\cos^6 x - 3\cos^4 x$$

$$= 3\left(\sin^4 x - \cos^4 x\right) - 4\left(\sin^6 x - \cos^6 x\right)$$

$$= 3\left(\sin^2 x - \cos^2 x\right)\left(\sin^2 x + \cos^2 x\right)$$

$$-4\left(\sin^2 x - \cos^2 x\right)\left(\sin^4 x + \sin^2 x\cos^2 x + \cos^4 x\right)$$

$$= -3\cos 2x + 4\cos 2x\left[1 - \sin^2 x\cos^2 x\right]$$

$$= -3\cos 2x + 4\cos 2x\left(1 - \frac{1}{4}\sin^2 2x\right)$$

$$= \cos 2x\left[-3 + 4\left(1 - \frac{1}{4}\sin^2 2x\right)\right]$$

$$= \cos 2x\left(1 - \sin^2 2x\right)$$

$$= \cos^3 2x$$

Cách 2 :

Ta có : $\sin 3x.\sin^3 x + \cos 3x.\cos^3 x$

$$= \sin 3x\left(\frac{3\sin x - \sin 3x}{4}\right) + \cos 3x\left(\frac{3\cos x + \cos 3x}{4}\right)$$

$$= \frac{3}{4}\left(\sin 3x\sin x + \cos 3x\cos x\right) + \frac{1}{4}\left(\cos^2 3x - \sin^2 3x\right)$$

$$= \frac{3}{4}\cos(3x - x) + \frac{1}{4}\cos 6x$$

$$= \frac{1}{4}(3\cos 2x + \cos 3.2x)$$

$$= \frac{1}{4}(3\cos 2x + 4\cos^3 2x - 3\cos 2x)$$ (bỏ dòng này cũng được)

$$= \cos^3 2x$$

Bài 16 : Chứng minh : $\cos 12^o + \cos 18^o - 4\cos 15^o.\cos 21^o \cos 24^o = -\frac{\sqrt{3}+1}{2}$

Ta có : $\cos 12^o + \cos 18^o - 4\cos 15^o\left(\cos 21^o \cos 24^o\right)$

$$= 2\cos 15^o \cos 3^o - 2\cos 15^o\left(\cos 45^o + \cos 3^o\right)$$

$$= 2\cos 15^o \cos 3^o - 2\cos 15^o \cos 45^o - 2\cos 15^o \cos 3^o$$

$$= -2\cos 15^o \cos 45^o$$

$$= -\left(\cos 60^o + \cos 30^o\right)$$

$$= -\frac{\sqrt{3}+1}{2}$$

Bài 17 : Tính $P = \sin^2 50^o + \sin^2 70 - \cos 50^o \cos 70^o$

Ta có : $P = \frac{1}{2}\left(1 - \cos 100^o\right) + \frac{1}{2}\left(1 - \cos 140^o\right) - \frac{1}{2}\left(\cos 120^o + \cos 20^o\right)$

$$P = 1 - \frac{1}{2}\left(\cos 100^o + \cos 140^o\right) - \frac{1}{2}\left(-\frac{1}{2} + \cos 20^o\right)$$

$$P = 1 - \left(\cos 120^o \cos 20^o\right) + \frac{1}{4} - \frac{1}{2}\cos 20^o$$

$$P = \frac{5}{4} + \frac{1}{2}\cos 20^o - \frac{1}{2}\cos 20^o = \frac{5}{4}$$

Bài 18 : Chứng minh : $tg30^o + tg40^o + tg50^o + tg60^o = \frac{8\sqrt{3}}{3}\cos 20^o$

Áp dụng : $tga + tgb = \frac{\sin(a+b)}{\cos a \cos b}$

Ta có : $\left(tg50^o + tg40^o\right) + \left(tg30^o + tg60^o\right)$

$$= \frac{\sin 90^o}{\cos 50^o \cos 40^o} + \frac{\sin 90^o}{\cos 30^o \cos 60^o}$$

$$= \frac{1}{\sin 40^o \cos 40^o} + \frac{1}{\frac{1}{2}\cos 30^o}$$

$$= \frac{2}{\sin 80^o} + \frac{2}{\cos 30^o}$$

$$= 2\left(\frac{1}{\cos 10^o} + \frac{1}{\cos 30^o}\right)$$

$$= 2\left(\frac{\cos 30^o + \cos 10^o}{\cos 10^o \cos 30^o}\right)$$
$$= 4\frac{\cos 20^p \cos 10^o}{\cos 10^o \cos 30^o}$$
$$= \frac{8\sqrt{3}}{3}\cos 20^o$$

Bài 19 : Cho ΔABC, Chứng minh :

a/ $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

b/ $socA + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

c/ $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$

d/ $\cos^2 A + \cos^2 B + \cos^2 C = -2\cos A\cos B\cos C$

e/ $tgA + tgB + tgC = tgA.tgB.tgC$

f/ $\cot gA.\cot gB + \cot gB.\cot gC + \cot gC.\cot gA = 1$

g/ $\cot g\frac{A}{2} + \cot g\frac{B}{2} + \cot g\frac{C}{2} = \cot g\frac{A}{2}.\cot g\frac{B}{2}.\cot g\frac{C}{2}$

a/ Ta có : $\sin A + \sin B + \sin C = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} + \sin(A+B)$

$$= 2\sin\frac{A+B}{2}\left(\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right)$$
$$= 4\cos\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2} \quad \left(\text{do } \frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2}\right)$$

b/ Ta có : $\cos A + \cos B + \cos C = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} - \cos(A+B)$

$$= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} - \left(2\cos^2\frac{A+B}{2} - 1\right)$$
$$= 2\cos\frac{A+B}{2}\left[\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right] + 1$$
$$= -4\cos\frac{A+B}{2}\sin\frac{A}{2}\sin\left(-\frac{B}{2}\right) + 1$$
$$= 4\sin\frac{C}{2}\sin\frac{A}{2}\sin\frac{B}{2} + 1$$

c/ $\sin 2A\sin 2B + \sin 2C = 2\sin(A+B)\cos(A-B) + 2\sin C\cos C$

$$= 2\sin C\cos(A-B) + 2\sin C\cos C$$
$$= 2\sin C[\cos(A-B) - \cos(A+B)]$$
$$= -4\sin C\sin A\sin(-B)$$
$$= 4\sin C\sin A\sin B$$

d/ $\cos^2 A + \cos^2 B + \cos^2 C$

$$= 1 + \frac{1}{2}(\cos 2A + \cos 2B) + \cos^2 C$$

$= 1 + \cos(A + B)\cos(A - B) + \cos^2 C$

$= 1 - \cos C\left[\cos(A - B) - \cos C\right]$ do $\left(\cos(A + B) = -\cos C\right)$

$= 1 - \cos C\left[\cos(A - B) + \cos(A + B)\right]$

$= 1 - 2\cos C.\cos A.\cos B$

e/ Do $a + b = \pi - C$ nên ta có

$tg(A + B) = -tgC$

$\Leftrightarrow \dfrac{tgA + tgB}{1 - tgAtgB} = -tgC$

$\Leftrightarrow tgA + tgB = -tgC + tgAtgBtgC$

$\Leftrightarrow tgA + tgB + tgC = tgAtgBtgC$

f/ Ta có : cotg(A+B) = - cotgC

$\Leftrightarrow \dfrac{1 - tgAtgB}{tgA + tgB} = -\cot gC$

$\Leftrightarrow \dfrac{\cot gA \cot gB - 1}{\cot gB + \cot gA} = -\cot gC$ (nhân tử và mẫu cho cotgA.cotgB)

$\Leftrightarrow \cot gA \cot gB - 1 = -\cot gC \cot gB - \cot gA \cot gC$

$\Leftrightarrow \cot gA \cot gB + \cot gB \cot gC + \cot gA \cot gC = 1$

g/ Ta có : $tg\dfrac{A + B}{2} = \cot g\dfrac{C}{2}$

$\Leftrightarrow \dfrac{tg\dfrac{A}{2} + tg\dfrac{B}{2}}{1 - tg\dfrac{A}{2}tg\dfrac{B}{2}} = \cot g\dfrac{C}{2}$

$\Leftrightarrow \dfrac{\cot g\dfrac{A}{2} + \cot g\dfrac{B}{2}}{\cot g\dfrac{A}{2}.\cot g\dfrac{B}{2} - 1} = \cot g\dfrac{C}{2}$ (nhân tử và mẫu cho cotg$\dfrac{A}{2}$.cotg$\dfrac{B}{2}$)

$\Leftrightarrow \cot g\dfrac{A}{2} + \cot g\dfrac{B}{2} = \cot g\dfrac{A}{2}\cot g\dfrac{B}{2}\cot g\dfrac{C}{2} - \cot g\dfrac{C}{2}$

$\Leftrightarrow \cot g\dfrac{A}{2} + \cot g\dfrac{B}{2} + \cot g\dfrac{C}{2} = \cot g\dfrac{A}{2}.\cot g\dfrac{B}{2}.\cot g\dfrac{C}{2}$

Bài 20 : Cho ΔABC. Chứng minh :

cos2A + cos2B + cos 2C + 4cosAcosBcosC + 1 = 0

Ta có : (cos2A + cos2B) + (cos2C + 1)

= 2 cos (A + B)cos(A - B) + 2cos^{2}C

= - 2cosCcos(A - B) + 2cos^{2}C

= - 2cosC[cos(A – B) + cos(A + B)] = - 4cosAcosBcosC

Do đó : cos2A + cos2B + cos2C + 1 + 4cosAcosBcosC = 0

Bài 21 : Cho ΔABC. Chứng minh :

$$\cos 3A + \cos 3B + \cos 3C = 1 - 4\sin\frac{3A}{2}\sin\frac{3B}{2}\sin\frac{3C}{2}$$

Ta có : $(\cos 3A + \cos 3B) + \cos 3C$

$$= 2\cos\frac{3}{2}(A+B)\cos\frac{3}{2}(A-B) + 1 - 2\sin^2\frac{3C}{2}$$

Mà : $A + B = \pi - C$ nên $\frac{3}{2}(A+B) = \frac{3}{2}\pi - \frac{3C}{2}$

$$\Rightarrow \cos\frac{3}{2}(A+B) = \cos\left(\frac{3\pi}{2} - \frac{3C}{2}\right)$$

$$= -\cos\left(\frac{\pi}{2} - \frac{3C}{2}\right)$$

$$= -\sin\frac{3C}{2}$$

Do đó : $\cos 3A + \cos 3B + \cos 3C$

$$= -2\sin\frac{3C}{2}\cos\frac{3(A-B)}{2} - 2\sin^2\frac{3C}{2} + 1$$

$$= -2\sin\frac{3C}{2}\left[\cos\frac{3(A-B)}{2} + \sin\frac{3C}{2}\right] + 1$$

$$= -2\sin\frac{3C}{2}\left[\cos\frac{3(A-B)}{2} - \cos\frac{3}{2}(A+B)\right] + 1$$

$$= 4\sin\frac{3C}{2}\sin\frac{3A}{2}\sin\left(\frac{-3B}{2}\right) + 1$$

$$= -4\sin\frac{3C}{2}\sin\frac{3A}{2}\sin\frac{3B}{2} + 1$$

Bài 22 : A, B, C là ba góc của một tam giác. Chứng minh :

$$\frac{\sin A + \sin B - \sin C}{\cos A + \cos B - \cos C + 1} = \operatorname{tg}\frac{A}{2}\operatorname{tg}\frac{B}{2}\operatorname{cotg}\frac{C}{2}$$

Ta có : $$\frac{\sin A + \sin B - \sin C}{\cos A + \cos B - \cos C + 1} = \frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2} - 2\sin\frac{C}{2}\cos\frac{C}{2}}{2\cos\frac{A+B}{2}\cos\frac{A-B}{2} + 2\sin^2\frac{C}{2}}$$

$$= \frac{2\cos\frac{C}{2}\left[\cos\frac{A-B}{2} - \sin\frac{C}{2}\right]}{2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} + \sin\frac{C}{2}\right]} = \operatorname{cotg}\frac{C}{2}\cdot\frac{\cos\frac{A-B}{2} - \cos\frac{A+B}{2}}{\cos\frac{A-B}{2} + \cos\frac{A+B}{2}}$$

$$= \operatorname{cotg}\frac{C}{2}\cdot\frac{-2\sin\frac{A}{2}.\sin\left(-\frac{B}{2}\right)}{2\cos\frac{A}{2}.\cos\frac{B}{2}}$$

$$=\cot g\frac{C}{2}.tg\frac{A}{2}.tg\frac{B}{2}$$

Bài 23 : Cho ΔABC. Chứng minh :

$$\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+\sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2}+\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}$$
$$=\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+tg\frac{A}{2}tg\frac{B}{2}+tg\frac{B}{2}tg\frac{C}{2}+tg\frac{A}{2}tg\frac{C}{2}(*)$$

Ta có : $\frac{A+B}{2}=\frac{\pi}{2}-\frac{C}{2}$ vậy $tg\left(\frac{A}{2}+\frac{B}{2}\right)=\cot g\frac{C}{2}$

$$\Leftrightarrow \frac{tg\frac{A}{2}+tg\frac{B}{2}}{1-tg\frac{A}{2}tg\frac{B}{2}}=\frac{1}{tg\frac{C}{2}}$$

$$\Leftrightarrow \left[tg\frac{A}{2}+tg\frac{B}{2}\right]tg\frac{C}{2}=1-tg\frac{A}{2}tg\frac{B}{2}$$

$$\Leftrightarrow tg\frac{A}{2}tg\frac{C}{2}+tg\frac{B}{2}tg\frac{C}{2}+tg\frac{A}{2}tg\frac{B}{2}=1(1)$$

Do đó : $(*)\Leftrightarrow \sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+\sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2}+\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}$

$$=\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+1 \text{ (do (1))}$$

$$\Leftrightarrow \sin\frac{A}{2}\left[\cos\frac{B}{2}\cos\frac{C}{2}-\sin\frac{B}{2}\sin\frac{C}{2}\right]+\cos\frac{A}{2}\left[\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{B}{2}\right]=1$$

$$\Leftrightarrow \sin\frac{A}{2}\cos\frac{B+C}{2}+\cos\frac{A}{2}\sin\frac{B+C}{2}=1$$

$\Leftrightarrow \sin\frac{A+B+C}{2}=1 \quad \Leftrightarrow \sin\frac{\pi}{2}=1$ (hiển nhiên đúng)

Bài 24 : Chứng minh : $tg\frac{A}{2}+tg\frac{B}{2}+tg\frac{C}{2}=\frac{3+\cos A+\cos B+\cos C}{\sin A+\sin B+\sin C}(*)$

Ta có :

$$\cos A+\cos B+\cos C+3=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}+\left[1-2\sin^2\frac{C}{2}\right]+3$$
$$=2\sin\frac{C}{2}\cos\frac{A-B}{2}+4-2\sin^2\frac{C}{2}$$
$$=2\sin\frac{C}{2}\left[\cos\frac{A-B}{2}-\sin\frac{C}{2}\right]+4$$
$$=2\sin\frac{C}{2}\left[\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right]+4$$
$$=4\sin\frac{C}{2}\sin\frac{A}{2}.\sin\frac{B}{2}+4 \ (1)$$

$$\sin A+\sin B+\sin C=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+\sin C$$
$$=2\cos\frac{C}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}$$
$$=2\cos\frac{C}{2}\left[\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right]$$
$$=4\cos\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}\ (2)$$

Từ (1) và (2) ta có :

$$(*)\Leftrightarrow\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}+\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}=\frac{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+1}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$\Leftrightarrow\sin\frac{A}{2}\left[\cos\frac{B}{2}\cos\frac{C}{2}\right]+\sin\frac{B}{2}\left[\cos\frac{A}{2}\cos\frac{C}{2}\right]+\sin\frac{C}{2}\left[\cos\frac{A}{2}\cos\frac{B}{2}\right]$$
$$=\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+1$$

$$\Leftrightarrow\sin\frac{A}{2}\left[\cos\frac{B}{2}\cos\frac{C}{2}-\sin\frac{B}{2}\sin\frac{C}{2}\right]+\cos\frac{A}{2}\left[\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{B}{2}\right]=1$$

$$\Leftrightarrow\sin\frac{A}{2}.\cos\frac{B+C}{2}+\cos\frac{A}{2}\sin\frac{B+C}{2}=1$$

$$\Leftrightarrow\sin\left[\frac{A+B+C}{2}\right]=1$$

$$\Leftrightarrow\sin\frac{\pi}{2}=1$$ (hiển nhiên đúng)

Bài 25 : Cho ΔABC. Chứng minh: $\dfrac{\sin\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}+\dfrac{\sin\frac{B}{2}}{\cos\frac{C}{2}\cos\frac{A}{2}}+\dfrac{\sin\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}=2$

Cách 1 :

Ta có : $$\frac{\sin\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}+\frac{\sin\frac{B}{2}}{\cos\frac{C}{2}\cos\frac{A}{2}}=\frac{\sin\frac{A}{2}\cos\frac{A}{2}+\sin\frac{B}{2}\cos\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$=\frac{1}{2}\frac{\sin A+\sin B}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$=\frac{\cos\frac{C}{2}.\cos\frac{A-B}{2}}{\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}}=\frac{\cos\left(\frac{A-B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}$$

Do đó : Vế trái $= \dfrac{\cos\left(\dfrac{A-B}{2}\right)}{\cos\dfrac{A}{2}\cos\dfrac{B}{2}} + \dfrac{\sin\dfrac{C}{2}}{\cos\dfrac{A}{2}\cos\dfrac{B}{2}} = \dfrac{\cos\dfrac{A-B}{2} + \cos\dfrac{A+B}{2}}{\cos\dfrac{A}{2}\cos\dfrac{B}{2}}$

$$= \dfrac{2\cos\dfrac{A}{2}\cos\dfrac{B}{2}}{\cos\dfrac{A}{2}\cos\dfrac{B}{2}} = 2$$

<u>Cách 2 :</u>

Ta có vế trái $= \dfrac{\cos\dfrac{B+C}{2}}{\cos\dfrac{B}{2}\cos\dfrac{C}{2}} + \dfrac{\cos\dfrac{A+C}{2}}{\cos\dfrac{C}{2}\cos\dfrac{A}{2}} + \dfrac{\cos\dfrac{A+B}{2}}{\cos\dfrac{A}{2}\cos\dfrac{B}{2}}$

$$= \dfrac{\cos\dfrac{B}{2}\cos\dfrac{C}{2} - \sin\dfrac{B}{2}\sin\dfrac{C}{2}}{\cos\dfrac{B}{2}\cos\dfrac{C}{2}} + \dfrac{\cos\dfrac{A}{2}\cos\dfrac{C}{2} - \sin\dfrac{A}{2}\sin\dfrac{C}{2}}{\cos\dfrac{C}{2}\cos\dfrac{A}{2}} + \dfrac{\cos\dfrac{A}{2}\cos\dfrac{B}{2} - \sin\dfrac{A}{2}\sin\dfrac{B}{2}}{\cos\dfrac{A}{2}\cos\dfrac{B}{2}}$$

$$= 3 - \left[\text{tg}\dfrac{B}{2}\text{tg}\dfrac{C}{2} + \text{tg}\dfrac{A}{2}\text{tg}\dfrac{C}{2} + \text{tg}\dfrac{A}{2}\text{tg}\dfrac{B}{2}\right]$$

Mà : $\text{tg}\dfrac{A}{2}\text{tg}\dfrac{B}{2} + \text{tg}\dfrac{B}{2}\text{tg}\dfrac{C}{2} + \text{tg}\dfrac{A}{2}\text{tg}\dfrac{B}{2} = 1$

(đã chứng minh tại bài 10)

Do đó : Vế trái = 3 – 1 = 2

<u>Bài 26 :</u> Cho ΔABC. Có $\cot g\dfrac{A}{2}, \cot g\dfrac{B}{2}, \cot g\dfrac{C}{2}$ theo tứ tự tạo cấp số cộng.

Chứng minh $\cot g\dfrac{A}{2}.\cot g\dfrac{C}{2} = 3$

Ta có : $\cot g\dfrac{A}{2}, \cot g\dfrac{B}{2}, \cot g\dfrac{C}{2}$ là cấp số cộng

$$\Leftrightarrow \cot g\dfrac{A}{2} + \cot g\dfrac{C}{2} = 2\cot g\dfrac{B}{2}$$

$$\Leftrightarrow \dfrac{\sin\dfrac{A+C}{2}}{\sin\dfrac{A}{2}\sin\dfrac{C}{2}} = \dfrac{2\cos\dfrac{B}{2}}{\sin\dfrac{B}{2}}$$

$$\Leftrightarrow \frac{\cos\frac{B}{2}}{\sin\frac{A}{2}\sin\frac{C}{2}} = \frac{2\cos\frac{B}{2}}{\sin\frac{B}{2}}$$

$$\Leftrightarrow \frac{1}{\sin\frac{A}{2}\sin\frac{C}{2}} = \frac{2}{\cos\frac{A+C}{2}} \text{ (do } 0<B<\pi \text{ nên } \cos\frac{B}{2}>0\text{)}$$

$$\Leftrightarrow \frac{\cos\frac{A}{2}\cos\frac{C}{2} - \sin\frac{A}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}.\sin\frac{C}{2}} = 2 \Leftrightarrow \cot g\frac{A}{2}\cot g\frac{C}{2} = 3$$

Bài 27 : Cho ΔABC. Chứng minh :

$$\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} = \frac{1}{2}\left[tg\frac{A}{2} + tg\frac{B}{2} + tg\frac{C}{2} + \cot g\frac{A}{2} + \cot g\frac{B}{2} + \cot g\frac{C}{2}\right]$$

Ta có : $\cot g\frac{A}{2} + \cot g\frac{B}{2} + \cot g\frac{C}{2} = \cot g\frac{A}{2}.\cot g\frac{B}{2}.\cot g\frac{C}{2}$

(Xem chứng minh bài 19g)

Mặt khác : $tg\alpha + \cot g\alpha = \frac{\sin\alpha}{\cos\alpha} + \frac{\cos\alpha}{\sin\alpha} = \frac{2}{\sin 2\alpha}$

Do đó : $\frac{1}{2}\left[tg\frac{A}{2} + tg\frac{B}{2} + tg\frac{C}{2} + cotg\frac{A}{2} + cotg\frac{B}{2} + cotg\frac{C}{2}\right]$

$$= \frac{1}{2}\left[tg\frac{A}{2} + tg\frac{B}{2} + tg\frac{C}{2}\right] + \frac{1}{2}\left[cotg\frac{A}{2} + cotg\frac{B}{2} + cotg\frac{C}{2}\right]$$

$$= \frac{1}{2}\left[tg\frac{A}{2} + \cot g\frac{A}{2}\right] + \frac{1}{2}\left[tg\frac{B}{2} + \cot g\frac{B}{2}\right] + \frac{1}{2}\left[tg\frac{C}{2} + \cot g\frac{C}{2}\right]$$

$$= \frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C}$$

BÀI TẬP

1. Chứng minh :

a/ $\cos\frac{\pi}{5} - \cos\frac{2\pi}{5} = \frac{1}{2}$

b/ $\frac{\cos 15^\circ + \sin 15^\circ}{\cos 15^\circ - \sin 15^\circ} = \sqrt{3}$

c/ $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$

d/ $\sin^3 2x \sin 6x + \cos^3 2x.\cos 6x = \cos^3 4x$

e/ $tg20^\circ.tg40^\circ.tg60^\circ.tg80^\circ = 3$

f/ $tg\frac{\pi}{6} + tg\frac{2\pi}{9} + tg\frac{5\pi}{18} + tg\frac{\pi}{3} = \frac{8\sqrt{3}}{3}\cos\frac{\pi}{9}$

g/ $\cos\frac{\pi}{15}.\cos\frac{2\pi}{15}.\cos\frac{3\pi}{15}.\cos\frac{4\pi}{15}.\cos\frac{5\pi}{15}.\cos\frac{6\pi}{15}.\cos\frac{7\pi}{15} = \frac{1}{2^7}$